Normal modes in a closed form aircraft dynamic model

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Abstract. In this work we look at aircraft flight from a dynamical systems perspective. Our foundation is a recently proposed closed form nonlinear model for the pitch plane or longitudinal motions of an aircraft. Exploiting a time scale separation inherent in the problem, we (a) show how an engineering flight dynamics model reduces to the Lanchester-Zhukovsky (LZ) glider model in the appropriate limit, (b) propose an extended version of the LZ model to include thrust and movable horizontal stabilizer, (c) obtain an explicit relation between the stabilizer deflection and the trimmed airspeed, and (d) derive fully explicit algebraic approximations of the short period and phugoid modes. Further, by combining aircraft with dynamical systems, we potentially introduce the technical aspects of airplane motion to a wide audience.

Keywords: Flight dynamics, Lanchester-Zhukovsky glider model, Short period mode, Phugoid mode

1 Introduction

Although flight dynamics and dynamical systems theory are well-established disciplines, the two have virtually zero overlap. A researcher in nonlinear dynamics outside of aerospace engineering typically has little idea of either the equations governing airplane motion or their solution structures. This probably happens because the Literature approach to flight dynamics draws heavily on data tables, obtained from experiments or computational fluid dynamics simulations. The data table dependence is common to both the linearized approach pioneered by George Bryan [1] and the nonlinear approach pioneered by Craig Jahnke [2], both approaches used in research papers and textbooks too numerous to cite. The data tables are aircraft-specific; further, for many aircraft such as Boeing 777 and Airbus A320, the tables are classified [3], [4]. To the best of our knowledge, only a few fighter jet models are public domain [5]–[7].

The only overlap between flight dynamics and more general dynamical systems theory is through the Lanchester-Zhukovsky (LZ) glider model, which describes the motions of an aerial vehicle having neither thrust nor any control over its pitch (note that a glider, as used in professional soaring, lacks thrust but does have pitch control). Letting V be the speed of the glider and η the angle which its flight path makes with the horizontal, the (non-dimensionalized) model equations are

$$\dot{V} = -\sin\eta - DV^2 \tag{1a}$$

$$V\dot{\eta} = -\cos\eta + V^2. \tag{1b}$$

Here the trigonometric terms represent the effect of gravity and the V^2 terms represent the effects of drag and lift. This equation, proposed independently in the late 19th to early 20th century by Frederick Lanchester [8] and Nikolai Yegorovich Zhukovsky [9], appears in a few textbooks on dynamical systems, for example Refs. [9], [10]. Among these, Ref. [9] presents a detailed stability and bifuraction analysis, finding quite a rich structure.

Equation (1) does not however make an appearance in the dedicated flight dynamics Literature, so the question of how or even whether the professional-grade models reduce to (1) in the appropriate limit is left unresolved. A relatively recent attempt [11] to derive one from the other appears contrived or even erroneous since the lift and drag coefficients are taken to depend on α, α is found to be sinusoidal and still the coefficients are treated as constants.

The demonstration of the reduction of an engineering flight dynamics model to the LZ glider model is one of the aims of the present work. The model we use is a fully explicit nonlinear equation of motion proposed by us last year [12], which replaces the data tables with closed form algebraic expressions. In the course of the derivation we also present an extension of the LZ model which includes thrust and movable horizontal stabilizer. Further, we obtain the algebraic approximations of the short period and phugoid modes in the model [12].

2 Closed form nonlinear model

The explicit nonlinear dynamical model proposed last year is briefly summarized in this Section. It features differential equations for five variables – the forward displacement of the aircraft y, the vertically upward displacement z, the speed V, the angle of elevation of the flight path η [i.e. $\eta = \arctan(V_z/V_y)$] and the pitch θ defined as the angle between the horizontal and the fuselage axis of symmetry. The angle of attack α is defined as $\alpha = \theta - \eta$. The equations feature two quantities which are inputted by the pilot (or autopilot) and can vary continuously – the thrust T and the downforce $\bar{f}_p$ applied at the tail. Related to the latter is the angle θ_E made by the tail with the horizontal (we have assumed the tail to consist of one movable piece). Among the parameters are the lift constants K_C and K_E for the wings and the tail (K is basically the constant of proportionality after the dependences on V^2 and α are factored out) and the drag constant C of the whole aircraft (the constant of proportionality after the V^2 dependence of parasitic drag is factored out). There are the mass of the aircraft m and the moment of inertia I about an axis passing through the centre of mass (CM). There are also the distances $\bar{d}_1$ and $\bar{d}_2$ between the aircraft CM and the centres of pressure (the point at which the lift effectively acts) of the wings and the tail, and $\bar{h}$, the distance between the fuselage symmetry axis and the line of action of the thrust. Some parameters have overhead bars : if for a typical airplane, quantity X happens to be negative as per our sign conventions then we define $\bar{X} = -X$ and use $\bar{X}$ in the equations so that the parameters there may be positive. Finally, we have a damping constant Γ for rotational motions. The geometry of the airplane as well as the forces acting on it are shown in Fig. 1.

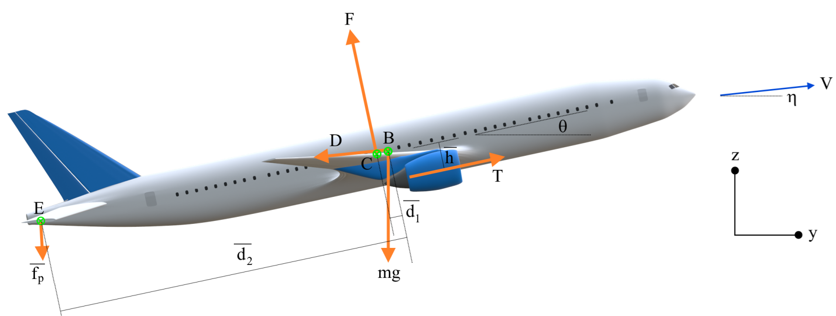

Fig. 1. Schematic diagram of the airplane showing the geometry as well as the forces on it. mg is the weight, acting at B, the aircraft CM. F is the aerodynamic force (lift plus induced drag) on the wings; it acts at C, their centre of pressure. D is the parasitic drag, assumed to act at B. T is the thrust. $\bar{f}_p$ is the aerodynamic force at the tail E.

The equation of motion obtained in Ref. [12] is shown in a schematic form in Fig. 2, where the significance of each term is briefly indicated. For more details including the derivation, we must refer to Ref. [12]. This figure will help us to keep track of the various terms as we analyse them in the next Section.

3 Stability analysis

3.1 Reformulation and simplification of the model equations

The original form of the model equation, shown in Fig. 2, treats as a fundamental quantity the force $\bar{f}_p$ commanded at the tail by the pilot. For stability analysis however it is customary to consider as fundamental the angle between the tail and the fuselage. This angle is called the deflection. Indeed, an uncontrolled glider as described by (1) perforce has the deflection constant, the value assigned during the design and manufacturing phase. As in Ref. [12] we consider an all-moving tail; we treat its deflection $\delta_E = -\bar{\delta}_E$ as a basic variable.

The relation between the deflection of the tail and the force applied on it is

$$\bar{f}_p = \frac{K_E V^2}{2} \sin 2\left(\eta - \theta + \bar{\delta}_E\right). \tag{2}$$

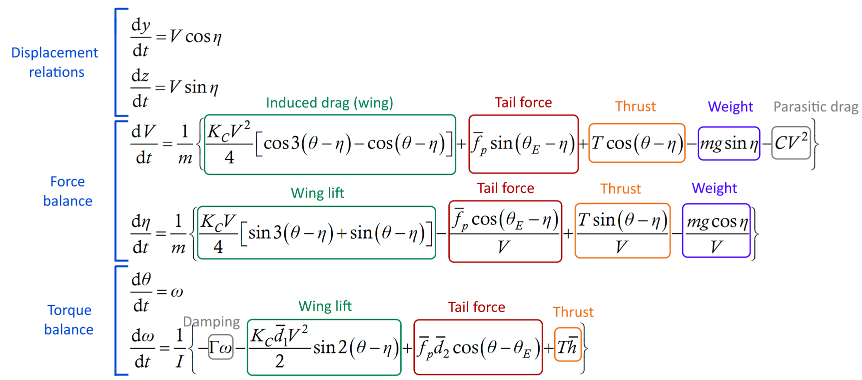

Fig. 2. Schematic representation of the equation of motion derived in Ref. [12].

Using this together with $\alpha = \theta - \eta$ in the last four equations of Fig. 2, we get

$$\dot{V} = \frac{1}{m}\left\{ \begin{array}{l} \dfrac{K_C V^2}{4}(\cos 3\alpha - \cos\alpha) + \dfrac{K_E V^2}{4}\left[\cos 3\left(\bar{\delta}_E - \alpha\right) - \cos\left(\bar{\delta}_E - \alpha\right)\right] + \\ T\cos\alpha - mg\sin\eta - CV^2 \end{array} \right\}, \tag{3a}$$

$$\dot{\eta} = \frac{1}{m}\left\{ \begin{array}{l} \dfrac{K_C V}{4}(\sin 3\alpha + \sin\alpha) - \dfrac{K_E V}{4}\left[\sin 3\left(\bar{\delta}_E - \alpha\right) + \sin\left(\bar{\delta}_E - \alpha\right)\right] + \\ \dfrac{T\sin\alpha}{V} - \dfrac{mg\cos\eta}{V} \end{array} \right\}, \tag{3b}$$

$$\dot{\theta} = \omega, \tag{3c}$$

$$\dot{\omega} = \frac{1}{I}\left\{ -\Gamma\omega - \frac{K_C \bar{d}_1 V^2}{2}\sin 2\alpha + \frac{K_E \bar{d}_2 V^2}{2}\sin 2\left(\bar{\delta}_E - \alpha\right)\cos\bar{\delta}_E + T\bar{h} \right\}. \tag{3d}$$

Now, our purpose in this Article is to get an overview of the airplane's dynamics rather than derive formulae having maximum accuracy. So we perform three simplification steps on the above equation. First we drop the terms featuring K_E in (3a),(3b). Because of the much larger area, the wings exert far larger forces than the tail. The K_E term is not ignorable in (3d) because $\bar{d}_2 >> \bar{d}_1$, in such a manner that the second and third terms in this RHS are of roughly equal size. Next we note that α and $\bar{\delta}_E - \alpha$ are the angles of attack at the wings and tail respectively; for the elements not to stall, the angles must be small. Accordingly we Taylor expand (3) to the lowest nonzero order in these quantities. Finally we ignore the term $T\bar{h}$ in (3d). In a typical jetliner the engines are only slightly below the fuselage centreline and their torque is a small fraction of those of the wings and tail. Moreover, thrust torque is not a significant factor influencing airplane dynamics, and our neglecting this term does not affect our fundamental understanding of them.

With these simplifications, (3) reads

$$\dot{V} = \frac{-K_C V^2 \alpha^2 - C V^2 + T - mg \sin\eta}{m}, \tag{4a}$$

$$\dot{\eta} = \frac{K_C V^2 \alpha + T\alpha - mg\cos\eta}{mV}, \tag{4b}$$

$$\dot{\theta} = \omega, \tag{4c}$$

$$\dot{\omega} = \frac{-\Gamma\omega - K_C \bar{d}_1 V^2 \alpha + K_E \bar{d}_2 V^2 \left(\bar{\delta}_E - \alpha\right)}{I}. \tag{4d}$$

This is the equation set which we shall analyse in the remainder of this Section.

3.2 Separation of time scales, fast dynamics

It so happens that, for many airplanes at least, α evolves much faster than V and η. Accordingly, we combine (4c) and (4d) to form an equation for $\ddot{\theta}$ and then subtract (4b) from the resultant to formulate an equation for $\ddot{\alpha}$. In that equation we set $\dot{V} = \dot{\eta} = 0$. This yields

$$\ddot{\alpha} + \left(\frac{\Gamma}{I} + \frac{K_C V}{m} + \frac{T}{mV}\right)\dot{\alpha} + \left(\frac{\left(K_C \bar{d}_1 + K_E \bar{d}_2\right) V^2}{I}\right)\alpha = \frac{K_E \bar{d}_2 V^2 \bar{\delta}_E}{I}, \tag{5}$$

which is a damped harmonic oscillator equation with an inhomogeneous term. The particular or steady state solution is

$$\alpha_p = \alpha^* = \frac{K_E \bar{d}_2}{K_C \bar{d}_1 + K_E \bar{d}_2}\bar{\delta}_E, \tag{6}$$

(we have introduced α^* here for a reason to be explained shortly). The homogeneous solutions are decaying oscillatory if

$$\frac{\Gamma}{I} + \frac{K_C V}{m} + \frac{T}{mV} < 2\left(\frac{K_C \bar{d}_1 + K_E \bar{d}_2}{I}\right)^{1/2} V, \tag{7}$$

and decaying non-oscillatory otherwise. In the oscillatory case, the characteristic exponents of the homogeneous solution are

$$\lambda_{1,2} = -\frac{1}{2}\left(\frac{\Gamma}{I} + \frac{K_C V}{m} + \frac{T}{mV}\right) \pm \mathrm{j}\frac{1}{2}\sqrt{\frac{4\left(K_C \bar{d}_1 + K_E \bar{d}_2\right) V^2}{I} - \left(\frac{\Gamma}{I} + \frac{K_C V}{m} + \frac{T}{mV}\right)^2}, \tag{8}$$

($\mathrm{j} = \sqrt{-1}$) while the exponents in the other case can be derived easily from (5).

We can see that the angle of attack gravitates to the particular solution $\alpha = \alpha_p$ in time. Since α_p is independent of V and η, we can identify it as α^*, the global fixed point of (4) [hence its introduction in (6)]. The homogeneous solutions of (5) depend on V; if we set $V = V^*$ (global fixed point) in (8), then $\lambda_{1,2}$ become two of the stability eigenvalues governing perturbations from the fixed point. The normal mode associated with α is called the short period mode; for the model (3), (5) is its algebraic approximation.

3.3 Slow dynamics

Having solved the fast system, we must now plug the solution into the slow one. In this case the fast system has a particularly simple solution – rapid approach of $\alpha(t)$ to the fixed point α^*. Hence, we substitute $\alpha = \alpha^*$ in (4a),(4b). Doing so we find

$$\dot{V} = \frac{-\left(K_C\alpha^{*2} + C\right)V^2 + T - mg\sin\eta}{m}, \tag{9a}$$

$$\dot{\eta} = \frac{K_C V^2\alpha^* + T\alpha^* - mg\cos\eta}{mV}. \tag{9b}$$

The first thing to note is that, if we define $D = K_C\alpha^{*2} + C$ and set $T = 0$ here, then we recover (1) [upto constants]. Thus we can think of (9) as the extension of the LZ model with thrust and movable stabilizer included. We have also formally demonstrated the reduction of the engineering flight dynamics model, Fig. 2 or (3), to the LZ model.

To find the fixed points of (9), we must solve (recall $D = K_C\alpha^{*2} + C$)

$$-DV^{*2} + T - mg\sin\eta^* = 0 \tag{10a}$$

$$K_C V^*\alpha^* + \frac{T\alpha^*}{V^*} - \frac{mg\cos\eta^*}{V^*} = 0. \tag{10b}$$

Some algebraic manipulations yield a quadratic for V^{*2} :

$$\left(K_C^2\alpha^{*2} + D^2\right)V^{*4} + \left(2K_C\alpha^{*2}T - 2DT\right)V^{*2} + T^2\left(1 + \alpha^{*2}\right) - m^2g^2 = 0, \tag{11}$$

which has the solution

$$V^* = \left(\frac{\left(D - K_C\alpha^{*2}\right)T \pm \sqrt{T^2(K_C\alpha^{*2} - D)^2 - (K_C^2\alpha^{*2} + D^2)\left(-m^2g^2 + T^2(1+\alpha^{*2})\right)}}{K_C^2\alpha^{*2} + D^2}\right)^{1/2}. \tag{12}$$

The positive sign turns out to be appropriate here. Given this, η^* can be easily obtained from either (10a) or (10b).

Now (12), though exact, is cumbersome in appearance. A much simper form results if we substitute α^* from (6) and then retain only the largest terms. This form can also be obtained from a static stability analysis of the model (3). We find

$$V^* = \sqrt{\frac{mg(K_C\bar{d}_1 + K_E\bar{d}_2)}{K_C K_E\bar{d}_2\bar{\delta}_E}}. \tag{13}$$

Substituting this into (10a) and assuming η^* to be small, we get for the trajectory elevation

$$\eta^* = -C\frac{K_C\bar{d}_1 + K_E\bar{d}_2}{K_C K_E\bar{d}_2\bar{\delta}_E} - \frac{K_E\bar{d}_2\bar{\delta}_E}{K_C\bar{d}_1 + K_E\bar{d}_2} + \frac{T}{mg}. \tag{14}$$

We have thus obtained explicit relations for the trimmed airspeed and climb angle in terms of the stabilizer deflection and the set power – relations which, though of considerable importance in aircraft operation, appear not to exist in Literature.

The fixed point analysis complete, we come now to the stability. Linearizing (9), we find that perturbations ΔV and $\Delta\eta$ from V^* and η^* satisfy the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\Delta V\\ \Delta\eta\end{bmatrix} = \frac{1}{m}\begin{bmatrix} -2DV^* & -mg\cos\eta^* \\ K_C\alpha^* + \frac{mg\cos\eta^* - T\alpha^*}{V^{*2}} & \frac{mg\sin\eta^*}{V^*}\end{bmatrix}\begin{bmatrix}\Delta V\\ \Delta\eta\end{bmatrix}, \tag{15}$$

whence the eigenvalues are

$$\lambda_{3,4} = \frac{1}{m}\left(-DV^* + \frac{mg\sin\eta^*}{2V^*} \pm \mathrm{j}\sqrt{\begin{array}{l}\frac{mg\cos\eta^*}{2}\left[K_C\alpha^* + \frac{mg\cos\eta^* - T\alpha^*}{V^{*2}}\right] \\ -2D^2V^{*2} - 3Dmg\sin\eta^* - \frac{m^2g^2\sin^2\eta^*}{2V^{*2}}\end{array}}\right). \tag{16}$$

These eigenvalues describe the phugoid mode; either (9) or (15) is its algebraic approximation depending on whether we seek a linear or nonlinear form.

4 Results

We now plot the modal eigenvalues as functions of speed. For this, we import the parameter values from Ref. [12], which are given in Table 1. As already stated in Ref. [12], these values are realistic for a largish narrow body airliner like Airbus A321 but do not actually correspond to the parameters for any one particular airplane. The values of m and T given in Table 1 are the maximum value which they can attain – lower values are of course possible.

Table 1. Parameter values for the model plane which we shall use here.

Parameter	m	g	K_C	K_E	T	$\bar{d}_1$	$\bar{d}_2$	$\bar{h}$	I	γ
SI Unit Value	10^5	9.8	1500	150	3×10^5	1	25	0.5	$64m$	$192m$

We consider a fixed elevation of $\eta^* = 6°$ and V^* ranging from 60 to 200 m/s (the entire speed range requires less than the maximum thrust) and show the short period and phugoid eigenvalues in Figs. 3 and 4. The short period mode has a period of a couple of seconds and damps out rapidly while the phugoid mode has a period of minutes and damps very slowly. Both of these are in agreement with what is known for real aircraft [13], [14]. In the phugoidal eigenvalue, a strong $1/V^*$ trend can be seen by comparing the values at $V^* = 100$ and $V^* = 200$ m/s; this is consistent with Lanchester's pioneering but basic analysis [8].

To conclude this calculation it remains only to verify that the separation of time scales which we assumed at the beginning holds true. We note that the

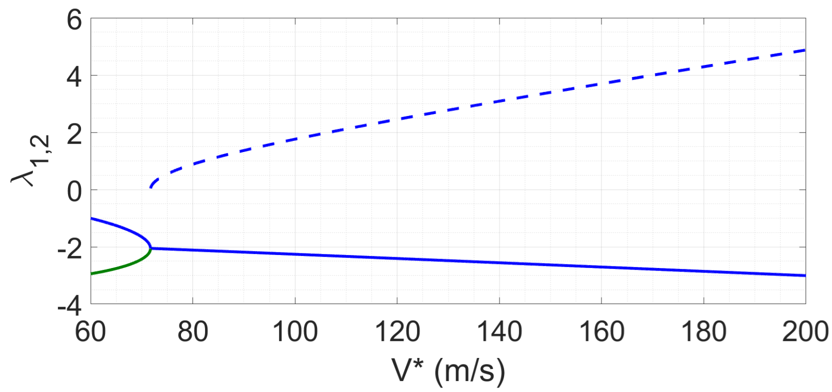

Fig. 3. Short period eigenvalue as V^* is varied, for η^* fixed at 6°. Upto a speed of 72 m/s, both the eigenvalues are real. The solid blue line denotes one of them and the solid green line the other. Beyond 72 m/s, the eigenvalues are a complex pair. In this regime, the solid blue line shows the real part of one of them and the dashed blue line the imaginary part. Since complex eigenvalues come in conjugate pairs, the other one perforce has the same real part and the negative imaginary part.

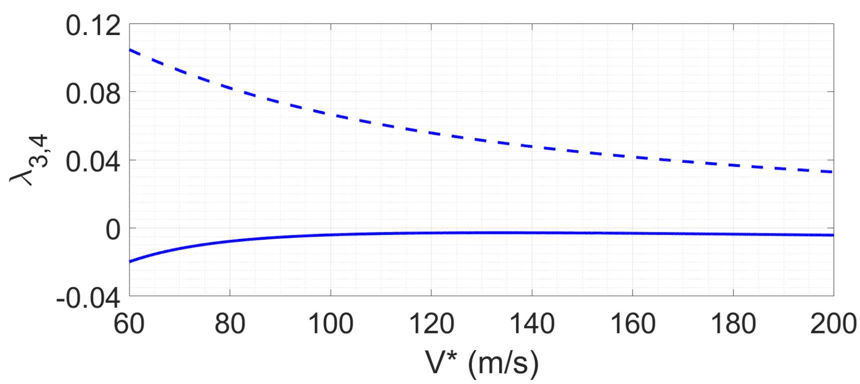

Fig. 4. Phugoid mode eigenvalue as V^* is varied, for η^* fixed at 6°. The solid line shows the real part and the dashed line the imaginary part of one eigenvalue. Since complex eigenvalues come in conjugate pairs, the other one perforce has the same real part and the negative imaginary part.

y-axis scale in Fig. 3 is 50 times greater than in Fig. 4, so our assumption is prima facie bona fide.

5 Discussion and conclusion

In this Article, we have exploited a recently proposed explicit nonlinear model of an aircraft to analyse flight dynamics from a nonlinear science perspective. This analysis has yielded a few results which are novel even in the field-specific Literature. An example is the nonlinear phugoid equation (9). Although it contains just a couple more terms beyond (1), I have never seen an explicit augmentation of that equation to include thrust and controllable elevator – the two elements which separate a jetliner from a dummy glider. Also novel are the relations (13) and (14) connecting the thrust and stabilizer deflection to the trimmed airspeed and climb angle.

The time scale separation is of course well known in engineering Literature. The derivation of algebraic (often called "literal") expressions for the normal modes usually starts from the linearized equations and then exploits the time scale separation. Only recently however [15] was this derivation was given in a full and rigorous manner. With respect to the short period mode, the stiffness of that mode was traditionally taken to be different from the change in moment with respect to angle of attack; this was corrected in Ref. [14]. In our model, the two are identical from the outset. With respect to the phugoid mode, reference [15] was the first to recognize that when the $\Delta\alpha$ (defined as $\alpha - \alpha^*$) equation is written in a sufficiently general way, then $\Delta\alpha$ does not decay in time to zero (as assumed in almost one hundred years of aerospace Literature) but to a constant (at the fast level) which depends on ΔV. This constant must then be accounted for while writing the $\Delta V, \Delta\eta$ system. In this Article, the simplifications made in going from (3) to (4) resulted in the $\Delta\alpha$ residual being zero (α^* is independent of V^* and η^*). A more accurate treatment of (3) would however have retained this residual in the nonlinear phugoid equation (9). It is in my opinion a significant virtue of our explicit aircraft model that it automatically yields the correct expressions for the various normal modes without our having to take special care to retain or drop certain terms.

The demonstration of the reduction of a professional-grade aircraft dynamic model to the LZ glider model is again a novel contribution of this Article. While the process here was smooth, the steps involved were not trivial. The simplification of (3) to (4) was a straightforward removal of smaller contributions but the rest hinged on two crucial factors : (a) the time scales of the $\alpha, \dot{\alpha}$ and V, η subsystems being widely separated, and (b) the solution for α being a constant independent of V and η. If any one of these happened not to hold true, then (1) would not have been a plausible reduced order model for (4).

In addition to proposing these novel results, our Article has the potential to introduce aircraft dynamics to the wider nonlinear dynamics community. Future work involves constructing the three-dimensional version of (3) and analysing it to characterize all five normal modes. This can lead to considerable insight regarding the relation between the aircraft design parameters and the stability eigenvalues. Further, given that the LZ model itself has a rich bifuraction structure [9], the full three-dimensional aircraft model should be a very interesting nonlinear dynamical system.

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